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Periodically time-modulated bistable systems: Nonstationary statistical properties

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The relaxation properties of a one-dimensional overdamped system modulated by an external periodic force are studied analytically by means of a perturbation approach. The validity of the approximations introduced is discussed in detail. The nonstationary nature of the process is illustrated by evaluating explicitly the autocorrelation function for the relaxation in a bistable potential. The predictions thus obtained are shown to compare favorably with the results of analogue simulation for the case of a quartic double-well potential. The stochastic resonance mechanism is proven to set in only when the periodic perturbation breaks the symmetry of the bistable potential.

I. INTRODUCTION

The interest for the study of bistable systems forced by a time-dependent perturbation[1,2] has been stimulated by the prediction that the output signal from a stochastic bistable system may be modulated in time by applying an external periodic forcing term.[3–5] The interplay of intrinsic noise and periodic driving mechanism may result in a sharp enhancement of the signal power spectrum within a narrow range about the forcing frequency. This phenomenon was explained by Benzi *et al.*[3,4] by relating the forcing frequency with the switch rate (Kramer's rate) of the unperturbed system between the stable states. To distinguish their finding from the well-known dynamical resonance,[6] Benzi *et al.*[4] introduced the notion of *stochastic resonance* (SR).

The existence of this effect has been confirmed by both numerical[3,4] and analogue simulation.[7–9] Its importance for application to a variety of physical systems driven by periodic forces is clearly illustrated by Roy and coworkers[9,10] who detected for the first time SR behaviors in a real physical system. Theoretical work, instead, focused on the dynamical mechanism underlying the SR in the attempt at reproducing the outcome of either the experiment or the simulation.[9,11,12]

Recently Jung and Hänggi[13] pointed out another important feature of stochastic systems driven by periodic forces. The systems are described by a nonstationary process, the relaxation dynamics of which is not strongly mixing.[14] This implies that important preparation effects may be detected under appropriate conditions.

The present paper is mainly aimed at discussing the nonstationary properties of stochastic relaxation in an overdamped bistable symmetric potential forced by a sinusoidal external force. In Sec. II we introduce a perturbation approach which is expected to describe the system fairly closely in the limit when the perturbation intensity is small compared to the noise intensity. Contrary to previous work based on the adiabatic approximation, our predictions apply also for the case of high forcing frequency, provided that the activation energy is large with respect to the noise intensity. The signal autocorrelation function is shown to depend on the preparation of the system at the initial time, thus proving the nonmixing nature of the process. In Sec. III we specialize the results of Sec. II for the case where the periodic modulation is coupled additively. A SR behavior is obtained in good agreement with the analog simulation. In Sec. IV we discuss an example of multiplicative modulation similar to that introduced by Moss *et al.*[15] No SR is induced by a period perturbation which preserves the symmetry of the potential. Our conclusions are drawn in Sec. V.

II. A PERTURBATION APPROACH

In this section we introduce a perturbation approach to the problem described by the nonlinear Langevin equation

$$\dot{x} = -V'(x,t) + \xi(t) , \tag{2.1}$$

where the prime and overdot denote x and t derivation, respectively. $\xi(t)$ is a zero-mean-valued Gaussian noise with autocorrelation function (ACF)

$$\langle \xi(t)\xi(0) \rangle = 2D\delta(t) . \tag{2.2}$$

The drift term in (2.1) can be separated into a deterministic force and a time-dependent perturbation according to the decomposition

$$V(x,t) = V(x) + P(x,t) . \tag{2.3}$$

Let us assume for simplicity that the potential $V(x)$ is bistable and symmetric and that the perturbation $P(x,t)$ is a sinusoidal function of t:

$$P(x,t) = -Ah(x)\cos(\Omega t + \theta) . \tag{2.4}$$

Here, Ω is the forcing frequency, $h(x)$ characterizes the nature of the coupling between the modulation and the process $x(t)$, and A is the perturbation parameter. The probability distribution function $\rho(x,t)$ of the process (2.1) obeys the Fokker-Planck equation (FPE)

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{\partial}{\partial x}\left[V'(x,t) + D\frac{\partial}{\partial x} \right]\rho(x,t) , \tag{2.5}$$

with boundary conditions $\lim_{|x|\to\infty}\rho(x,t)=0$.

The relevant statistical properties are entirely described by the above FPE. Indeed, all we need to determine is the probability density $\rho(x,t)$ and the conditional probability $p(x',t';x,t)$, which, in turn, can be expressed in terms of the eigenfunctions and eigenvalues of the unperturbed problem.[16] Alternatively, we can proceed as follows. Let us define

$$\Psi(x,t)=e^{V(x,t)/2D}\rho(x,t) \ . \tag{2.6}$$

On substituting $\rho(x,t)$ with $\Psi(x,t)$, Eq. (2.5) can be rewritten as

$$-\frac{\partial}{\partial t}\Psi(x,t)=-D\frac{\partial^2}{\partial^2 x}\Psi(x,t)+\left[\frac{1}{4D}V'(x,t)^2-\tfrac{1}{2}V''(x,t)-\frac{1}{2D}\dot{V}(x,t)\right]\Psi(x,t) \ . \tag{2.7}$$

The starting FPE, (2.5), may be mapped into the Schrödinger equation (SE) corresponding to the Hamiltonian[16]

$$\hat{H}=-D\frac{\partial^2}{\partial x^2}+W(x,t) \ , \tag{2.8}$$

with

$$W(x,t)=\frac{1}{4D}V'(x,t)^2-\tfrac{1}{2}V''(x,t)-\frac{1}{2D}\dot{V}(x,t) \ . \tag{2.9}$$

Making use of definitions (2.3) and (2.4) we separate $\hat{H}$ into an unperturbed operator, $\hat{H}^0_-$ and a perturbation $\hat{H}^1$, i.e.,

$$\hat{H}=\hat{H}^0_-+\hat{H}^1 \ , \tag{2.10}$$

with

$$\hat{H}^0_-=-D\frac{d^2}{dx^2}+\frac{1}{4D}V'(x)^2-\frac{1}{2}V''(x) \tag{2.11a}$$

and

$$\hat{H}^1=\frac{1}{4D}[P'(x,t)^2+2P'(x,t)V'(x)-2\dot{P}(x,t)-2DP''(x,t)] \ . \tag{2.11b}$$

Let us assume that the solution to the eigenvalue problem associated with $\hat{H}^0_-$

$$\hat{H}^0_-\varphi_n^-(x)=E_n^-\varphi_n^-(x) \tag{2.12}$$

is known. $\Psi(x,t)$ can be expanded on the complete orthonormal set of eigenfunctions $\{\varphi_n^-\}$:

$$\Psi(x,t)=\sum_k c_k(t)\varphi_k^-(x)e^{-E_k^- t} \ . \tag{2.13}$$

The initial state $\Psi(x,0)$ is given assigning the values of the coefficients $c_k \equiv c_k(0)$ with the only condition of normalization

$$\int dx\,\rho(x,0)=\sum_k c_k\int dx\,e^{-V(x,0)/2D}\varphi_k^-(x)=1 \ . \tag{2.14}$$

From now on the x integrations are meant to be taken over any compact domain of the x axis where the process $x(t)$ has been localized. One should keep in mind that by construction $E_0^-=0$ and $\varphi_0^-(x)^2$ coincide with the stationary probability distribution of the unperturbed problem (2.5) with $P=0$ [see Eq. (2.6)].

The conditional probability $p(x',t';x,t)$ can be mapped into the relevant propagation kernel for $\Psi(x,t)$,

$$K(x',t';x,t)=e^{V(x',t')/2D}p(x',t';x,t)e^{-V(x,t)/2D} \ . \tag{2.15}$$

$K(x',t'x,t)$ can be expanded in powers of $\hat{H}^1$ according to a well-known formula due to Feynman[17]

$$K(x',t';x,t)=K^0(x',t';x,t)-\int_t^{t'}dt_1\int dx_1 K^0(x',t',x_1,t_1)\hat{H}^1(x_1,t_1)K^0(x_1,t_1;x,t)$$
$$+\int_t^{t'}dt_1\int dx_1\int_t^{t_1}dt_2\int dx_2 K^0(x',t';x_1,t_1)\hat{H}^1(x_1,t_1)K^0(x_1,t_1;x_2,t_2)\hat{H}^1(x_2,t_2)K^0(x_2,t_2;x,t)+\cdots \ . \tag{2.16}$$

The unperturbed kernel $K^0(x',t';x,t)$ can also be expanded on the basis $\{\varphi_n^-\}$, i.e.,

$$K^0(x',t';x,t)=\sum_k\varphi_k^-(x')\varphi_k^-(x)e^{-E_k^-(t'-t)} \ . \tag{2.17}$$

Substituting (2.17) in (2.16) and collecting only the terms up to the second order in the parameter A yield the following approximation to the propagation kernel:

$$K(x',t';x,t)=K^0(x',t';x,t)+\frac{A}{2D}K^1(x',t';x,t)-\frac{A^2}{4D}K^2(x',t';x,t)+\frac{A^2}{4D^2}K^3(x',t';x,t)+O(A^3) \ , \tag{2.18}$$

where $K^i(x',t';x,t)$ with $i=1,2,3$ are expressible analytically in terms of $\{\varphi_n^-\}$ and $\{E_n^-\}$. We report the details of our calculations in the Appendix.

A. Calculation of the probability density $\rho(x,t)$

We are now in the position to determine the time evolution of the probability density $\rho(x,t)$. Let us assume that the periodic perturbation $P(x,t)$ is switched on at $t_0=0+$ with the unperturbed system arbitrarily disturbed

$$\rho(x,0)=N_0\sum_k c_k\varphi_0^-(x)\varphi_k^-(x) \qquad (2.19)$$

[condition (2.14) implies $c_0N_0=1$]. At a generic time $t>0$ the probability density is given by

$$\rho(x,t)=\int dy\, p(x,t;y,0)\rho(y,0)\ . \qquad (2.20)$$

Due to the transformation (2.6), Eq. (2.20) can be rewritten as

$$\rho(x,t)=\int dy\, e^{-V(x,t)/2D}K(x,t;y,0)\Psi(y,0)\ . \qquad (2.21)$$

On expanding the exponential function of the integral (2.21) in powers of the parameter A and making use of the expansion (2.18) for the propagation kernel, we arrive at the analytical expression for $\rho(x,t)$, (A14), reported in the Appendix. A few properties of that involved expression are remarkable.

(i) $\rho(x,t)$ exhibits a transient behavior described by the terms containing $e^{-E_n^- t}$ with $n>0$. In order to simplify our discussion we have assumed that the spectrum $\{E_n^-\}$ is discrete and that $E_1^- \ll E_2^-$, E_n^- being ordered according to their magnitude. This amounts to requiring that $V(x)$ diverges for $|x|\to\infty$ at least as $|x|^\alpha$ with $\alpha>1$, and that the potential barrier between the two minima, ΔV, is large compared with the noise intensity D.

(ii) If we take the limit $t\to\infty$ we immediately recognize that $\rho_\infty(x,t)=\lim_{t\to\infty}\rho(x,t)$ oscillates in time with frequency $\nu_\Omega=\Omega/2\pi$, as it should be. More importantly, $\rho(x,t)$ retains information of the initial state preparation $\rho(x,t)$ even at asymptotically large times [as shown by the dependence of $\rho_\infty(x,t)$ on the coefficients c_k with $k>0$ and on the phase θ of the modulation].

(iii) For the purpose of comparison with the results of analogue simulation one must average $\rho_\infty(x,t)$ over one forcing period $T_\Omega=2\pi/\Omega$ and, as the quantity thus obtained still depends on θ, over a uniform distribution of the phase of the perturbation (2.4). The averaged asymptotic expression for the signal distribution $\rho_\infty(x)$ is reported in Eq. (A18) of the Appendix.

B. Calculation of the autocorrelation function $C(\tau)$

Due to the nonstationary nature of the process (2.1) and (2.2) the definition of the $x(t)$ ACF might present some ambiguity. The definition we propose is summarized in the following scheme: The periodic modulation is switched on at time $t_0=0+$ with a certain phase θ, (2.4), the unperturbed system being at equilibrium with distribution $\rho(x)=\varphi_0^-(x)^2$ (i.e., $c_0=N_0^{-1}$ and $c_k=0$ for $k\neq0$). The perturbed system is made propagate in the presence of the perturbation up to the time t; the two-time correlation of the process $x(t)$ is then defined as

$$\langle x(t+\tau)x(t)\rangle=\int dy\int dz\, zyp(z,t+\tau;y,t)\rho(y,t). \qquad (2.22)$$

On employing Eqs. (2.6) and (2.21) we cast (2.22) in the form of a triple integral

$$\langle x(t+\tau)x(t)\rangle = \int dx\int dy\int dz\, zye^{-V(x,t+\tau)/2D}K(z,t+\tau;y,t) \times K(y,t;x,0)\Psi(x,0)\ . \qquad (2.23)$$

After some lengthy and tedious algebraic manipulations we obtained a perturbation expansion of (2.23) accurate up to the second order in A. Owing to our preparation procedure, the result for $\langle x(t+\tau)x(t)\rangle$ exhibits transient terms which can be eliminated by making t tend to infinity. Of course $\lim_{t\to\infty}\langle x(t+\tau)x(t)\rangle$ is still a periodic function of t. It should be remembered here that the process under study is nonstationary. The dependence on the initial time can be eliminated by averaging the initial state over one forcing period T_Ω. The result thus obtained is denoted by $C(\tau)$. $C(\tau)$ depends on the phase θ. We report in the Appendix our evaluation of $C(\tau)$ after the average over the angle θ has been taken.

Expression (A19) for $C(\tau)$ simplifies when (i) we neglect the exponentially vanishing terms corresponding to the eigenvalues E_n^- with $n\geq2$, in agreement with the assumption that $E_1^-\ll E_2^-$; (ii) we make use of the fact that the perturbed potential $V(x)$ is a symmetric function of x and the parity of the eigenfunctions $\varphi_n^-(x)$, (2.12), coincides with the parity of the relevant index n; (iii) we neglect all of the remaining matrix elements containing eigenfunctions with index $n>1$. For convenience we distinguish two cases.

(1) The case of odd coupling $h(x)$. In this case we have

$$C(\tau)=\left[Z_0^u-\frac{A^2}{8D^2}Z_1^u\right]e^{-\lambda|\tau|}+\frac{A^2}{8D^2}[Z_2^u\cos(\Omega\tau)+Z_3^u\sin(\Omega|\tau|)]\ , \qquad (2.24)$$

where $\lambda\equiv E_1^-$ and

$$Z_0^u=\langle0|x|1\rangle^2\ , \qquad (2.25a)$$

$$Z_1^u=\frac{3\lambda^2-\Omega^2}{(\lambda^2+\Omega^2)^2}(\langle0|V'h'-Dh''|1\rangle^2+\Omega^2\langle0|h|1\rangle^2)\langle0|x|1\rangle^2 + \frac{1}{\lambda^2+\Omega^2}(\lambda\langle0|V'h'-Dh''|1\rangle+\Omega^2\langle0|h|1\rangle)\langle0|xh|0\rangle\langle0|x|1\rangle-\tfrac{1}{2}\langle0|xh^2|1\rangle\langle0|x|1\rangle\ , \qquad (2.25b)$$

$$Z_2^u=\frac{2\lambda^2}{(\lambda^2+\Omega^2)^2}(\langle 0|V'h'-Dh''|1\rangle^2+\Omega^2\langle 0|h|1\rangle^2)\langle 0|x|1\rangle^2+\frac{2\lambda}{\lambda^2+\Omega^2}\langle 0|V'h'-Dh''|1\rangle\langle 0|xh|0\rangle\langle 0|x|1\rangle\ , \tag{2.25c}$$

$$Z_3^u=\frac{2\lambda\Omega}{(\lambda^2+\Omega^2)^2}(\langle 0|V'h'-Dh''|1\rangle^2+\Omega^2\langle 0|h|1\rangle^2)\langle 0|x|1\rangle^2-\frac{2\lambda\Omega}{\lambda^2+\Omega^2}\langle 0|h|1\rangle\langle 0|xh|0\rangle\langle 0|x|1\rangle\ .$$

(2) The case of even coupling $h(x)$. In this case we have

$$C(\tau)=\left[Z_0^g-\frac{A^2}{8D^2}Z_1^g\right]e^{-\lambda|\tau|}+\frac{A^2}{8D^2}\left[Z_2^g\cos(\Omega\tau)+Z_3^g\sin(\Omega|\tau|)\right]e^{-\lambda|\tau|}\ , \tag{2.26}$$

where

$$Z_0^g=\langle 0|x|1\rangle^2\ , \tag{2.27a}$$

$$Z_1^g=\frac{1}{\Omega^2}[\langle 0|V'h'-Dh''|0\rangle\langle 1|V'h'-Dh''|1\rangle-\langle 0|V'h'-Dh''|0\rangle^2-\langle 1|V'h'-Dh''|1\rangle^2$$
$$+\Omega^2(\langle 0|h|0\rangle\langle 1|h|1\rangle-\langle 0|h|0\rangle^2-\langle 1|h|1\rangle^2)]\langle 0|x|1\rangle^2+\langle 1|h|1\rangle\langle 0|xh|1\rangle\langle 0|x|1\rangle-\tfrac{1}{2}\langle 0|xh^2|1\rangle\langle 0|x|1\rangle\ , \tag{2.27b}$$

$$Z_2^g=\frac{1}{\Omega^2}[\langle 1|V'h'-Dh''|1\rangle(\langle 0|V'h'-Dh''|0\rangle-\langle 1|V'h'-Dh''|1\rangle)$$
$$+\Omega^2\langle 1|h|1\rangle(\langle 0|h|0\rangle-\langle 1|h|1\rangle)]\langle 0|x|1\rangle^2+(\langle 1|h|1\rangle-\langle 0|h|0\rangle)\langle 0|xh|1\rangle\langle 0|x|1\rangle\ , \tag{2.27c}$$

$$Z_3^g=\frac{1}{\Omega}[\langle 1|h|1\rangle(\langle 0|V'h'-Dh''|0\rangle-\langle 1|V'h'-Dh''|1\rangle)-\langle 1|V'h'-Dh''|1\rangle(\langle 0|h|0\rangle-\langle 1|h|1\rangle)]\langle 0|x|1\rangle^2$$
$$+\frac{1}{\Omega}(\langle 1|V'h'-Dh''|1\rangle-\langle 0|V'h'-Dh''|0\rangle)\langle 0|xh|1\rangle\langle 0|x|1\rangle\ . \tag{2.27d}$$

The bracket $\langle m|f|n\rangle$ replaces here $\int dx\, \varphi_m^-(x)f(x)\rho_n^-(x)$ for any function $f(x)$.

On comparing (2.24) and (2.26) it is apparent that $C(\tau)$ may exhibit an oscillating behavior for asymptotically large τ only when $h(x)$ *is not* an even function of x. The presence of oscillatory terms with frequency ν_Ω in the expression (2.24) for $C(\tau)$ has been observed experimentally by revealing a deltalike spike at the forcing frequency in the power spectrum of the signal $x(t)$.[7,8]

Finally, it should be noticed that our definition of signal ACF differs from that adopted in Ref. 13, where no sudden switch from the unperturbed stationary state to the nonstationary state $\rho_\infty(x,t)$ at time $t_0=0^+$ is introduced.

III. THE CASE OF ODD COUPLING

We address now the case of odd coupling (2.24) with special reference to the additive modulation

$$P(x,t)=-Ax\cos(\Omega t+\theta)\ , \tag{3.1}$$

i.e., $h(x)=x$. We rewrite the coefficients Z_i^u of Eq. (2.25) in a more explicit form as follows:

$$Z_0^u=\langle 0|x|1\rangle^2\ , \tag{3.2a}$$

$$Z_1^u=\frac{3\lambda^2-\Omega^2}{(\lambda^2+\Omega^2)^2}(\langle 0|V'|1\rangle^2+\Omega^2\langle 0|x|1\rangle^2)\langle 0|x|1\rangle^2$$
$$+\frac{1}{\lambda^2+\Omega^2}(\lambda\langle 0|V'|1\rangle+\Omega^2\langle 0|x|1\rangle)$$
$$\times\langle 0|x^2|0\rangle\langle 0|x|1\rangle-\tfrac{1}{2}\langle 0|x^3|1\rangle\langle 0|x|1\rangle\ , \tag{3.2b}$$

$$Z_2^u=\frac{2\lambda^2}{(\lambda^2+\Omega^2)^2}(\langle 0|V'|1\rangle^2+\Omega^2\langle 0|x|1\rangle^2)\langle 0|x|1\rangle^2$$
$$+\frac{2\lambda}{\lambda^2+\Omega^2}\langle 0|V'|1\rangle\langle 0|x^2|0\rangle\langle 0|x|1\rangle\ , \tag{3.2c}$$

$$Z_3^u=\frac{2\lambda\Omega}{(\lambda^2+\Omega^2)^2}(\langle 0|V'|1\rangle^2+\Omega^2\langle 0|x|1\rangle^2)\langle 0|x|1\rangle^2$$
$$-\frac{2\lambda\Omega}{\lambda^2+\Omega^2}\langle 0|x^2|0\rangle\langle 0|x|1\rangle^2\ , \tag{3.2d}$$

where $\langle 0|x^2|0\rangle$ represents the second moment of the stationary distribution $\rho(x)$ of the unperturbed system. The calculation of the integrals $\langle 0|x^3|1\rangle$, $\langle 0|x|1\rangle$, and $\langle 0|V'|1\rangle$, instead, requires that we determine first the eigenfunction $\varphi_1^-(x)$. This might be done with good accuracy by means of some computational code like the matrix-continued-fraction algorithm developed by Risken and co-workers.[16]

An accurate approximate analytical expression for $\varphi_1^-(x)$ can be obtained by means of supersymmetric quantum-mechanical techniques.[18] In Sec. II we have mapped the FPE into a SE corresponding to the Hamiltonian $\hat{H}$. The unperturbed system is then described by the Hamiltonian

$$\hat{H}^0_- = -D\frac{d^2}{dx^2} + \frac{1}{4D}V'(x)^2 - \frac{1}{2}V''(x) , \tag{3.3}$$

with discrete eigenvalues $\{E_n^-\}$ and eigenfunctions $\{\varphi_n^-\}$. $\hat{H}^0_-$ and its supersymmetric partner

$$\hat{H}^0_+ = -D\frac{d^2}{dx^2} + \frac{1}{4D}V'(x)^2 + \tfrac{1}{2}V''(x) \tag{3.4}$$

exhibit an interesting property: the eigenvalue problems associated with $\hat{H}^0_+$ and $\hat{H}^0_-$ are related by the equalities following:

$$E_n^- = E_{n-1}^+ , \quad n > 0 \tag{3.5}$$

$$\varphi_n^-(x) = \frac{1}{(E_n^-)^{1/2}} \hat{A}\varphi_{n-1}^+(x), \quad n > 0 \tag{3.6}$$

where $\{E_n^+\}$ and $\{\varphi_n^+\}$ are the eigenvalues and the normalized eigenfunctions of $\hat{H}^0_+$ and

$$\hat{A} = -\sqrt{D}\frac{d}{dx} + \frac{1}{2\sqrt{D}}V'(x) , \tag{3.7}$$

The ground state of $\hat{H}^0_-$ is determined by $E_0^- = 0$ and the eigenfunction

$$\varphi_0^-(x) = \frac{1}{N_0}e^{-V(x)/2D} , \tag{3.8}$$

with

$$N_0^2 = \int dx\, e^{-V(x)/D} .$$

Keung *et al.*[18] noticed that the function

$$\tilde{\varphi}_0^+(-x) = \tilde{\varphi}_0^+(x) = \frac{1}{\alpha}\frac{1}{\varphi_0^-(x)}\int_x^\infty dy[\varphi_0^-(y)]^2 \quad (x \geq 0) \tag{3.9}$$

where

$$\alpha = 2\int_0^\infty dx \left[\frac{1}{\varphi_0^-(x)}\int_x^\infty dy[\varphi_0^-(y)]^2\right]^2$$

is an eigenfunction of the operator

$$\tilde{H}^0_+ = \hat{H}^0_+ - 4D\varphi_0^-(0)\delta(x) , \tag{3.10}$$

with zero eigenvalue. On treating the second term on the right-hand side of Eq. (3.10) as a perturbation, one obtains the first-order approximation for E_0^+,

$$E_0^+ = \frac{D}{\alpha} , \tag{3.11}$$

while at the zeroth order for the corresponding eigenfunction $\varphi_0^+(x) = \tilde{\varphi}_0^+(x)$. The equality (3.6) leads to our estimate for $\varphi_1^-(x)$:

$$\varphi_1^-(x) = [2\theta(x) - 1]\varphi_0^-(x) , \tag{3.12}$$

where $\theta(x)$ denotes the step function. The approximations (3.11) and (3.12) are expected to be very accurate for small D values.

The calculation of the integrals contained in (3.2) is now straightforward. For comparison with the results of analogue simulation we specialize our computation for the case of a quartic double-well potential:

$$V(x) = \frac{\omega_0^2 x_m^2}{8}\left[\frac{x^2}{x_m^2} - 1\right]^2 . \tag{3.13}$$

In Fig. 1 we display the amplitude $C_{max}(D)$ for the oscillating component of the normalized ACF (2.24), $C(\tau)/C(0)$, for three values of the forcing frequency ν_Ω. As a remarkable feature of our approach we predict that the oscillating behavior of $C(\tau)$ tends to disappear both at large and small D. This makes quite a difference with the adiabatic approximation previously introduced in the literature.[2,9,11] We discuss this point in a companion paper[19] with reference to the phenomenon of SR.

In Fig. 1, we also reported the results of analog simulation for the same values of all the parameters involved. For details about the setup of our simulator we refer the reader to the following paper.[19] In order to appreciate the closeness of our predictions one should keep in mind that due to experimental limitations[19] we cannot produce accurate analogue determinations of $C_{max}(D)$ for $Ax_m/D < 0.4$. It is apparent from Eq. (2.24) that the second-order truncation would get more reliable for smaller values of Ax_m/D. The comparison between theoretical and analogue results is therefore quite satisfactory.

It should be remarked that the built-in averaging procedure of our wave-form analyzer (for details see Appendix of Ref. 20) does not allow one to appreciate the magnitude of most effects due to the nonstationary nature of the process under study. In particular, at the present state of our investigation, we cannot contrast our definition of signal ACF with that proposed in Ref. 13.

Finally, a peculiar feature of our preparation scheme (Sec. II A) is the prediction of a negative time dephasing of the oscillating component of the signal $x(t)$. In fact, the oscillating behavior of $C(\tau)$, (2.24), at large τ can be represented in the following form:

$$C(\tau) \sim \frac{A^2}{8D^2}(Z_2^u + Z_3^u)\cos(\Omega\tau + \phi) , \tag{3.14}$$

where

$$\phi = \arctan\left[-\frac{Z_3^u}{Z_2^u}\right] . \tag{3.15}$$

For the relatively simple system discussed in the present section, ϕ could be appreciable only for very small values of D, where the amplitude of $C(\tau)$, (3.14), is vanishingly small. On the other hand, for values of D which maximize the signal modulation, namely, $C_{max}(D)$, ϕ is expectedly too small to observe experimentally. All that can be readily seen on comparing (2.25c) with (2.25d). Z_2^u and Z_3^u are given, respectively, as the sum and the difference of two quantities of the same order of magnitude and, furthermore, Z_3^u/Z_2^u is roughly proportional to the forcing-frequency to escape-rate ratio, Ω/λ. No significant evidence for the existence of any signal de-

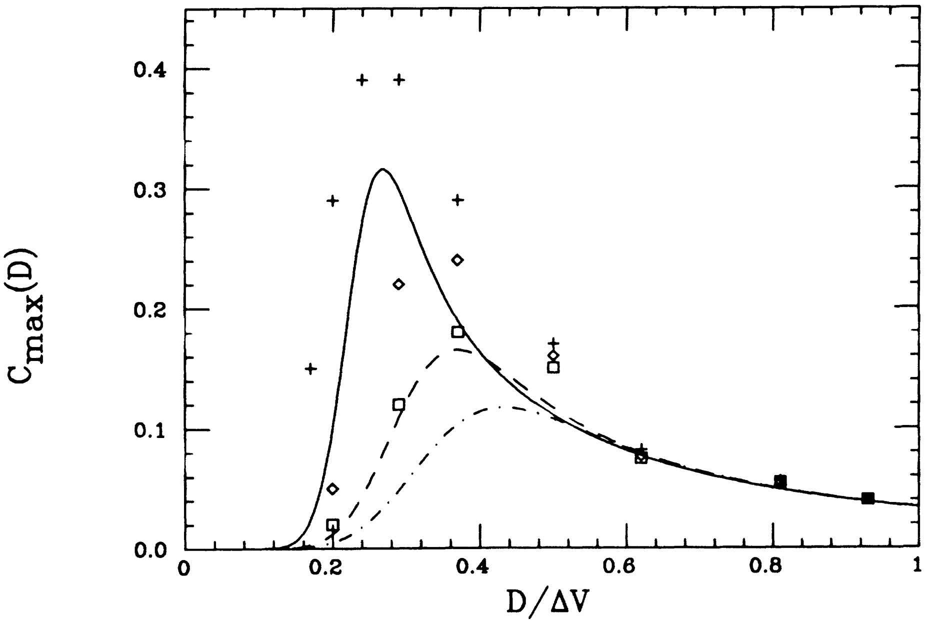

FIG. 1. $C_{max}(D)$ vs $D/\Delta V$ for different values of the forcing frequency. $V(x)$ is given in (3.13) with $x_m = 7.3$ V and $\omega_0 = 13.6$ kHz; the perturbation strength is $Ax_m/\Delta V = 0.5$. Curves represent our theoretical predictions for $\nu_\Omega = 10$ Hz (solid), $\nu_\Omega = 30$ Hz (dashed), and $\nu_\Omega = 50$ Hz (dot-dashed). The corresponding results of analog simulation are represented by crosses, lozenges, and squares, respectively.

phasing has been detected by means of the data analysis procedure employed in our simulation. No such dephasing is predicted in the thoery of Ref. 13.

IV. THE CASE OF EVEN COUPLING

We conclude our analysis of the process (2.1) and (2.2) with a few remarks about the case of even coupling. An example of this class of couplings has been introduced in Ref. 15. For simplicity we confine ourselves to the case $h(x) = \frac{1}{2}x^2$ (multiplicative modulation). We showed in Sec. II that $C(\tau)$, (2.26), vanishes for large values of τ, thus implying that no δ-like spike can be detected in the relevant power spectrum of the signal $x(t)$.

However, the simulation of Ref. 15 seems to suggest that spikes corresponding to the forcing frequencies can be observed for an even coupling also. In fact, the explanation of the effect revealed by Moss and co-workers is very simple. Let us take the Fourier transform of $C(\tau)$ in Eq. (2.26):

$$C(\omega) = \sqrt{2/\pi}\int_0^\infty C(\tau)\cos(\omega\tau)d\tau = \sqrt{2/\pi}\left[\left[Z_0^g - \frac{A^2}{8D^2}Z_1^g\right]\frac{\lambda}{\lambda^2+\omega^2} + \frac{A^2}{16D^2}\left[\frac{\lambda}{\lambda^2+(\omega-\Omega)^2} + \frac{\lambda}{\lambda^2+(\omega+\Omega)^2}\right]Z_2^g + \frac{A^2}{16D^2}\left[\frac{\Omega-\omega}{\lambda^2+(\omega-\Omega)^2} + \frac{\Omega+\omega}{\lambda^2+(\omega+\Omega)^2}\right]Z_3^g\right]. \quad (4.1)$$

The numerical evaluation of the coefficients Z_i^g $(i = 0,1,2,3)$ in (4.1) can be carried out following the approximate procedure of Sec. III.

$C(\omega)$ is dominated by a Lorentzian curve centered about the origin which describes the exponential decay of $C(\tau)$ at large times. For small values of the noise intensity, however, the second-order corrections may become important. In fact, we can determine D and Ω in such a way that $\Omega > \lambda$. For values of ω close to the forcing-frequency Ω the Fourier transform of the leading term of $C(\tau)$ may happen to be small enough to make the profile of the spectrum branch proportional to A^2 directly observable. It assumes the form of a Lorentzian curve peaked in Ω.

The analogue simulation of $C(\omega)$ for the potential of Eq. (3.13) at several values of the noise intensity confirms our predictions. On decreasing D, a peak shows up at $\omega = \Omega$. Since Z_2^g turns out to be quite insensitive to the value of D, we would expect that the height of the peak in Ω is proportional to D^{-2} and its width is of order of the

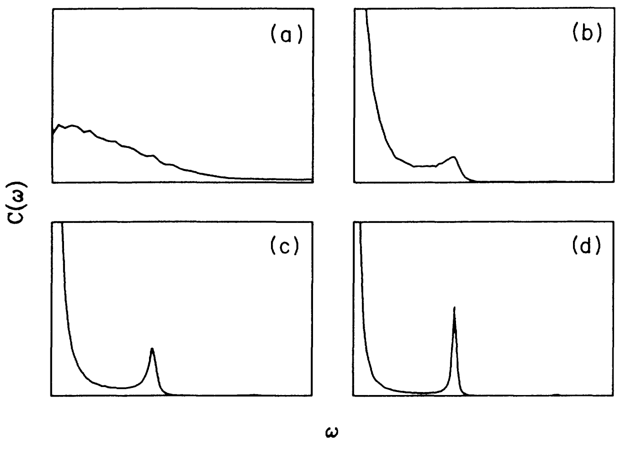

FIG. 2. $C(\omega)$ for different values of the noise intensity: (a) $D/\Delta V=0.5$, (b) $D/\Delta V=0.04$, (c) $D/\Delta V=0.03$, (d) $D/\Delta V=0.02$. The potential is as in Fig. 1. The parameters of the perturbation are kept constant: $Ax_m/\Delta V=0.5$ and $\nu_\Omega=30$ Hz.

relaxation rate λ. This behavior, too, is apparent from the spectra displayed in Fig. 2.

The peak at $\omega\simeq\Omega$ is precisely the spike revealed by the authors of Ref. 15. It is clear from the discussion above, however, that this peak cannot be described by a δ function and that its intensity blows up with vanishing D. Therefore no SR mechanism is induced by a periodic modulation that does not break the symmetry of the bistable potential.

V. CONCLUSIONS

We studied the stochastic relaxation in a bistable symmetric potential modulated periodically with time and subject to additive Gaussian noise. We have illustrated the nonstationary nature of such a process by calculating explicitly the signal distribution and ACF within a perturbation formalism inspired to quantum mechanics. The mechanism underlying SR is indeed a resonant one as clearly shown by the dependence of the oscillating tail of the signal ACF on the noise intensity. The modulation of the output signal induced by the external periodic perturbation is enhanced for forcing frequencies equal or close to the relevant escape rate in the driving potential. Finally, we have pointed out that SR is always related to the presence of a periodic perturbation mechanism which breaks the symmetry of the unperturbed system

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APPENDIX

The perturbation $\hat H^1$ comprises both linear and quadratic terms in A, i.e.,

$$\hat H^1 = -\frac{A}{2D}\hat H^1_1 + \frac{A^2}{4D}\hat H^1_2 , \tag{A1}$$

with

$$\hat H^1_1 = [h'(x)V'(x) - Dh''(x)]\cos(\Omega t+\theta) + \Omega h(x)\sin(\Omega t+\theta) , \tag{A2}$$

$$\hat H^1_2 = h'(x)^2\cos^2(\Omega t+\theta) . \tag{A3}$$

The corrections to the free propagation kernel up to the second order in A, (2.18), follow immediately,

$$K^i(x',t';x,t) = \sum_{m,n}\varphi_m^-(x')e^{-E_m^- t'} f^i_{mn}(t',t)\varphi_n^-(x)e^{E_n^- t} , \quad i=1,2,3 \tag{A4}$$

where

$$f^1_{mn}(t',t) = \int_t^{t'} dt_1 e^{(E_m^- E_n^-)t_1}\langle m|\hat H^1_1(t_1)|n\rangle = e^{(E_m^- - E_n^-)t'}F^1_{mn}(t') - e^{(E_m^- - E_n^-)t}F^1_{mn}(t) , \tag{A5}$$

$$f^2_{mn}(t',t) = \int_t^{t'} dt_1 e^{(E_m^- - E_n^-)t_1}\langle m|\hat H^1_2(t_1)|n\rangle = e^{(E_m^- - E_n^-)t'}F^2_{mn}(t') - e^{(E_m^- - E_n^-)t}F^2_{mn}(t) , \tag{A6}$$

$$\begin{aligned} f^3_{mn}(t',t) &= \sum_p \int_t^{t'} dt_1 e^{(E_m^- - E_p^-)t_1}\langle m|\hat H^1_1(t_1)|p\rangle f^1_{pn}(t_1,t) \\ &= e^{(E_m^- - E_n^-)t'}F^3_{mn}(t') - e^{(E_m^- - E_n^-)t}F^3_{mn}(t) + \sum_p [e^{(E_m^- - E_n^-)t}F^1_{mp}(t)F^1_{pn}(t) - e^{(E_m^- - E_p^-)t'}F^1_{mp}(t')e^{(E_p^- - E_n^-)t}F^1_{pn}(t)] . \end{aligned} \tag{A7}$$

$F^i_{mn}(t)$ are purely oscillating functions,

$$F^1_{mn}(t) = G^0_{mn}\cos(\Omega t+\theta) + G^1_{mn}\sin(\Omega t+\theta) , \tag{A8}$$

$$F^2_{mn}(t) = \tfrac{1}{2}\langle m|h'^2|n\rangle \left[\frac{1}{(E_m^- - E_n^-)} + \frac{(E_m^- - E_n^-)\cos[2(\Omega t+\theta)] + 2\Omega\sin[2(\Omega t+\theta)]}{(E_m^- - E_n^-)^2 + 4\Omega^2} \right] , \tag{A9}$$

$$F^3_{mn}(t)=\tfrac{1}{2}G^3_{mn}+\tfrac{1}{2}\sum_p\left[(\langle m|h'V'-Dh''|p\rangle G^0_{pn}-\langle m|h|n\rangle\Omega G^1_{pn})\frac{(E^-_m-E^-_n)\cos[2(\Omega+\theta)]+2\Omega\sin[2(\Omega t+\theta)]}{(E^-_m-E^-_n)^2+4\Omega^2}\right.$$
$$\left.+(\langle m|h'V'-Dh''|p\rangle G^1_{pn}+\langle m|h|n\rangle\Omega G^0_{pn})\frac{(E^-_m-E^-_n)\sin[2(\Omega t+\theta)]-2\Omega\cos[2(\Omega t+\theta)]}{(E^-_m-E^-_n)^2+4\Omega^2}\right], \tag{A10}$$

with

$$G^0_{mn}=\frac{\langle m|h'V'-Dh''|n\rangle(E^-_m-E^-_n)-\langle m|h|n\rangle\Omega^2}{(E^-_m-E^-_n)^2+\Omega^2}, \tag{A11}$$

$$G^1_{mn}=\frac{\langle m|h'V'-Dh''|n\rangle\Omega+\langle m|h|n\rangle\Omega(E^-_m-E^-_n)}{(E^-_m-E^-_n)^2+\Omega^2}, \tag{A12}$$

$$G^3_{mn}=\sum_p\frac{\langle m|h'V'-Dh''|p\rangle G^0_{pn}+\langle m|h|p\rangle\Omega G^1_{pn}}{E^-_m-E^-_n}. \tag{A13}$$

We employed an analogous procedure to calculate $\rho(x,t)$, perturbatively. On substituting (A4)–(A13) into (2.21) we obtain

$$\rho(x,t)=\rho^0(x,t)+\frac{A}{2D}\rho^1(x,t)+\frac{A^2}{4D^2}\rho^2(x,t)+O(A^3), \tag{A14}$$

where

$$\rho^0(x,t)=e^{-V(x)/2D}\sum_i c_i\varphi^-_i(x)e^{-E^-_i t}, \tag{A15}$$

$$\rho^1(x,t)=e^{-V(x)/2D}\left[h(x)\cos(\Omega t+\theta)\sum_i c_i\varphi^-_i(x)e^{-E^-_i t}+\sum_{i,j}c_j\varphi^-_i(x)[e^{-E^-_j t}F^1_{ij}(t)-e^{-E^-_i t}-F^1_{ij}(0)]\right], \tag{A16}$$

$$\rho^2(x,t)=e^{-V(x)/2D}\left[\tfrac{1}{2}h(x)^2\cos^2(\Omega t+\theta)\sum_i c_i\varphi^-_i(x)e^{-E^-_i t}\right.$$
$$+h(x)\cos(\Omega t+\theta)\sum_{i,j}c_j\varphi^-_i(x)[e^{-E^-_j t}F^1_{ij}(t)-e^{-E^-_i t}F^1_{ij}(0)]$$
$$-D\sum_{i,j}c_j\varphi^-_i(x)[e^{-E^-_j t}F^2_{ij}(t)-e^{-E^-_i t}F^2_{ij}(0)]$$
$$+\sum_{i,j}c_j\varphi^-_i(x)\left[e^{-E^-_j t}F^3_{ij}(t)-e^{-E^-_i t}F^3_{ij}(0)\right.$$
$$\left.\left.-\sum_k e^{-E^-_k t}F^1_{ik}(t)F^1_{kj}(0)+e^{-E^-_i t}\sum_k F^1_{ik}(0)F^1_{kj}(0)\right]\right]. \tag{A17}$$

$\rho(x,t)$ for t asymptotically large oscillates in time with frequency $\nu_\Omega=\Omega/2\pi$. Averaging over t and θ we obtain

$$\rho_\infty(x)=e^{-V(x)/2D}\left[c_0\varphi^-_0(x)+\frac{A^2}{8D^2}\left[\tfrac{1}{2}h(x)^2c_0\varphi^-_0(x)+h(x)c_0\sum_i\varphi^-_i(x)G^0_{i0}\right.\right.$$
$$\left.\left.+\varphi^-_0(x)\sum_{i,j}c_i(G^0_{0j}G^0_{ji}+G^1_{0j}G^0_{ji})+c_0\sum_i\varphi^-_i(x)G^3_{i0}-\varphi^-_0(x)\sum_i c_iG^3_{0i}\right]\right]. \tag{A18}$$

With analogous manipulations we can obtain a second-order expression for the ACF (2.23). Our results after averaging over t and θ reads

$$
\begin{aligned}
C(\tau)=\sum_i \langle 0|x|i\rangle^2 e^{-E_i^-|\tau|}+\frac{A^2}{8D^2}\Bigg\{ & \tfrac{1}{2}\sum_i \langle 0|xh^2|i\rangle\langle 0|x|i\rangle e^{-E_i^-|\tau|}+\sum_{i,j}\langle 0|xh|i\rangle\langle 0|x|j\rangle e^{-E_j^-|\tau|}G_{ij}^0 \\
& -\sum_{i,j}\langle 0|xh|i\rangle\langle 0|x|j\rangle e^{-E_i^-|\tau|}[G_{ij}^0\cos(\Omega\tau)-G_{ij}^1\sin(\Omega|\tau|)] \\
& +\sum_{i,j}\langle 0|xh|i\rangle\langle i|x|j\rangle e^{-E_i^-|\tau|}[G_{j0}^0\cos(\Omega\tau)-G_{j0}^1\sin(\Omega|\tau|)] \\
& +\sum_{i,j}\langle 0|x|i\rangle\langle 0|x|j\rangle(e^{-E_j^-|\tau|}-e^{-E_i^-|\tau|})G_{ij}^3 \\
& +\sum_{i,j}\langle 0|x|i\rangle\langle i|x|j\rangle e^{-E_i^-|\tau|}G_{j0}^3-\sum_i\langle 0|x|i\rangle^2 e^{-E_i^-|\tau|}G_{00}^3 \\
& -\sum_{i,j,k}\langle 0|x|i\rangle\langle 0|x|j\rangle e^{-E_k^-|\tau|} \\
& \qquad\times[(G_{ik}^0G_{kj}^0+G_{ik}^1G_{kj}^1)\cos(\Omega\tau)-(G_{ik}^0G_{kj}^1-G_{ik}^1G_{kj}^0)\sin(\Omega|\tau|)] \\
& +\sum_{i,j,k}\langle 0|x|i\rangle\langle 0|x|j\rangle e^{-E_i^-|\tau|}(G_{ik}^0G_{kj}^0+G_{ik}^1G_{kj}^1) \\
& +\sum_{i,k}\langle 0|x|i\rangle^2 e^{-E_i^-|\tau|}(G_{0k}^0G_{k0}^0+G_{0k}^1G_{k0}^1) \\
& +\sum_{i,j,k}\langle 0|x|i\rangle\langle j|x|k\rangle e^{-E_j^-|\tau|} \\
& \qquad\times[(G_{ij}^0G_{k0}^0+G_{ij}^1G_{k0}^1)\cos(\Omega\tau)-(G_{ij}^0G_{k0}^1-G_{ij}^1G_{k0}^0)\sin(\Omega|\tau|)] \\
& -\sum_{i,j,k}\langle 0|x|i\rangle\langle j|x|k\rangle e^{-E_i^-|\tau|}(G_{ij}^-G_{k0}^0+G_{ij}^1G_{k0}^1)\Bigg\}\ .
\end{aligned}
\tag{A19}
$$

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